Mechanics: Rotation

FIZIKA SPhO Training

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1 Notes

In real life, many objects are not well-modelled by point masses, since they have finite sizes. In this case, the point of application of forces matter, and will lead to rotation of the object.

1.1 Rotational Kinematics

Just like in regular translational kinematics, we will first study rotation without caring about where it comes from.

1.1.1 Rotational Analogues of the SUVAT Equations

In translational kinematics, we are concerned with the quantities x, v, a and t. We may draw an analogy to rotational quantities as follows:

$$x \longleftrightarrow \theta,\ v \longleftrightarrow \omega,\ a \longleftrightarrow \alpha \tag{1}$$

where θ is the angular displacement, ω is the angular velocity and α is the angular acceleration.

As a result, the following definitions hold:

$$\boldsymbol{\omega} := \frac{d\boldsymbol{\theta}}{dt} \tag{2}$$

$$\boldsymbol{\alpha} := \frac{d\boldsymbol{\omega}}{dt} \tag{3}$$

and the corresponding rotational analogues of the SUVAT equations are:

$$\boldsymbol{\omega}_f = \boldsymbol{\omega}_i + \boldsymbol{\alpha} t \tag{4}$$

$$\boldsymbol{\theta}_f = \boldsymbol{\theta}_i + \boldsymbol{\omega}_i t + \frac{1}{2}\boldsymbol{\alpha} t^2 \tag{5}$$

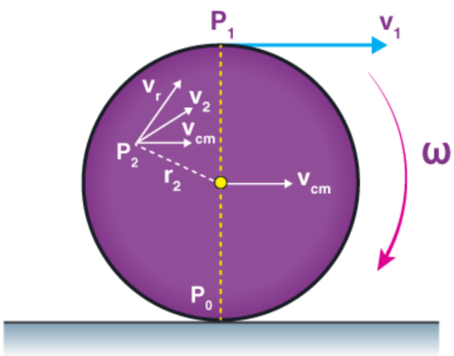
$$|\boldsymbol{\omega}_f|^2 = |\boldsymbol{\omega}_i|^2 + 2\boldsymbol{\alpha} \cdot (\boldsymbol{\theta}_f - \boldsymbol{\theta}_i) \tag{6}$$

1.1.2 Rolling

We commonly discuss rolling, which is a **combination of translation and rotation.** In other words, you may think of rolling as a sum of:

1. Translation of a fixed point on an object
2. Rotation of the object about that fixed point

For instance, for the ball below, the rolling motion is a sum of the translation of the CM and rotation about the CM.



For **rolling without slipping,** the condition is

$$v_{CM} = R\omega \tag{7}$$

while for **rolling with slipping,** we cannot simply relate v_{CM} and ω (i.e. $v_{CM} \neq R\omega$).

Differentiating Equation (7) with respect to time, we can relate a_{CM} and α for the non-slip case:

$$a_{CM} = R\alpha \tag{8}$$

Remark. When there is no slipping, friction is static and does not do work. When there is slipping, friction is kinetic and does work. You need to account for this if you want to calculate the frictional force or apply energy conservation!

1.1.3 Instantaneous Centre of Rotation (ICOR)

It can be more helpful to visualise rolling using the ICOR.

The velocity at any point (for instance, point P_2 in the diagram above) is given by

$$\mathbf{v}_{P_2} = \mathbf{v}_{trans} + \mathbf{v}_{rot} = \mathbf{v}_{CM} + \boldsymbol{\omega} \times \mathbf{r}_2 \tag{9}$$

Applying this equation, we can see that at point P_0, $\mathbf{v}_{CM}$ and $\boldsymbol{\omega} \times \boldsymbol{r}_0$ perfectly cancel out, **if there is no slipping.** (Both have magnitude $|\mathbf{v}_{CM}|$ and point in opposite directions.)

Hence, the point P_0 is actually *instantaneously* stationary!

$$\mathbf{v}_{P_0} = \mathbf{0} \tag{10}$$

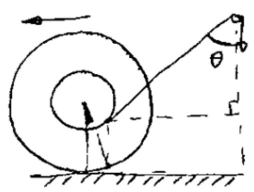
We usually call P_0 the ICOR. You can equivalently visualise rolling as a **fixed-axis rotation** about the ICOR.

Remark. This visualisation (the ICOR) and the visualisation of the translation of the CM + rotation about the CM are equivalent.

Another nice geometrical fact is that any point on the rigid body moves perpendicular to a line drawn to the ICOR. Using the diagram above,

$$P_0P_2 \perp \mathbf{v}_2 \tag{11}$$

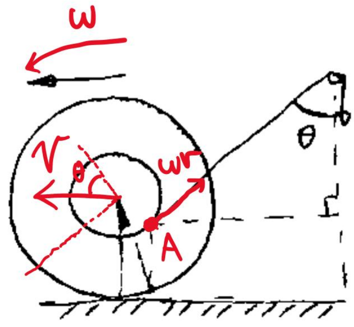
Example 1.1 (Ricardo)**.** An inextensible string is spooled around a cylinder at its inner radius r. The loose end of the string is rolled around a pulley and pulled with speed v. The string makes an angle θ with the vertical. If the outer radius of the cylinder is R, determine the CM speed of the cylinder, assuming it rolls without slipping.



Let the CM speed of the cylinder be V. Because it rolls without slipping,

$$V = R\omega$$

Now, let's draw all the relevant velocity vectors on the diagram.



Considering the velocities at point A, since point A moves with speed v along the string,

$$\omega r - V\sin\theta = v \quad \Rightarrow \quad V\frac{r}{R} - V\sin\theta = v \quad \Rightarrow \quad V = \frac{v}{\frac{r}{R} - \sin\theta}$$

Remark. This idea of taking a convenient point and summing velocity vectors is useful, especially when the said point has its velocity **constrained.** In this case, the constraint is the fact that the string is pulling point A, which forces it to have speed v along the string.

Example 1.2 (Kevin Zhou)**.** A bicycle wheel is rolling without slipping. When it is photographed, its spokes look blurred, except along a curve of special points. What is this curve?

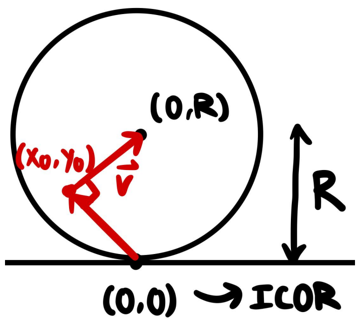
For the points to not be blurred, we want the points to be moving **parallel to the spokes.** This ensures that in the small time where the picture is captured, the point still lies along the spoke and will hence appear clear.

If $\mathbf{r}$ is the position vector from the CM, then

$$\mathbf{v} = \mathbf{v}_{CM} + \boldsymbol{\omega} \times \mathbf{r}$$

describes the velocity at $\mathbf{r}$. We also want the velocity to point along the spoke, so $\mathbf{v} \parallel \mathbf{r}$.

Quantitatively, let the wheel have radius R, and set up coordinates such that the centre of the wheel is at $(0, R)$ and the bottom of the wheel (which is also the ICOR, as the wheel is not slipping) is at $(0, 0)$. Consider the velocity of some point (x_0, y_0) on the curve.



The gradient between this point and the ICOR is $\frac{y_0}{x_0}$. From Equation (11), the velocity vector will be perpendicular to this line. The gradient of the normal line is $-\frac{x_0}{y_0}$.

We can find the equation of this normal line. Since it also passes through (x_0, y_0),

$$-\frac{x_0}{y_0} = \frac{y - y_0}{x - x_0} \quad \Rightarrow \quad y = -\frac{x_0}{y_0}x + \frac{x_0^2 + y_0^2}{y_0}$$

This line must pass through $(0, R)$. Thus,

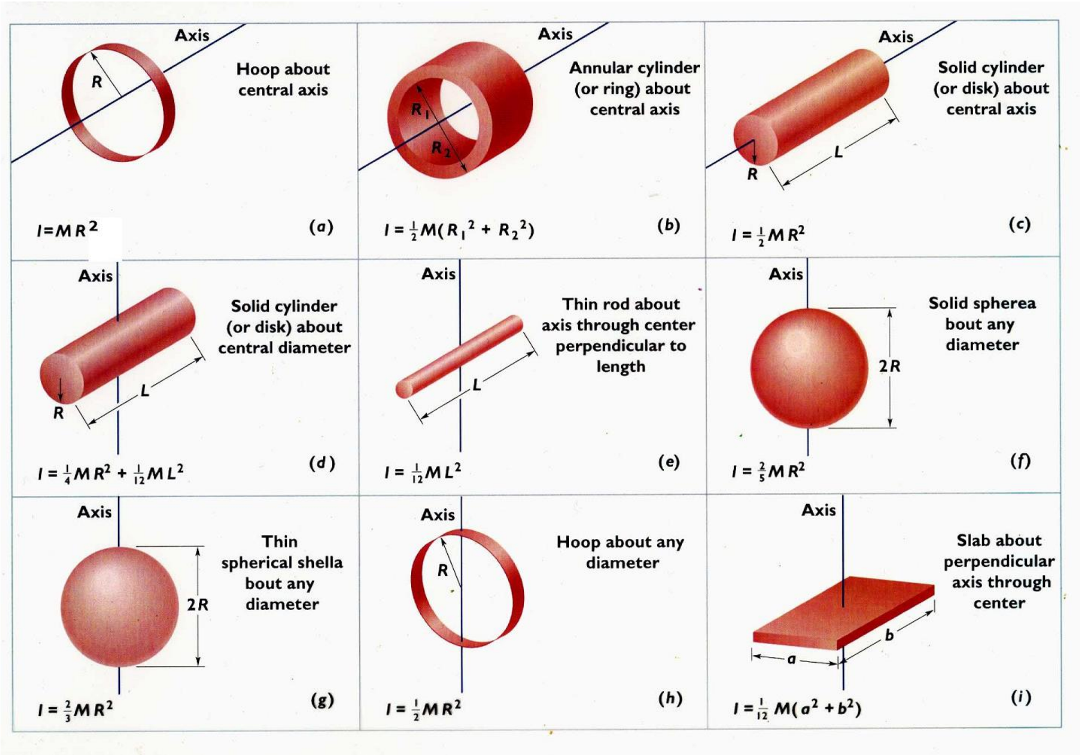
$$R = \frac{x_0^2 + y_0^2}{y_0} \quad \Rightarrow \quad x_0^2 + y_0^2 - Ry_0 = 0 \quad \Rightarrow \quad x_0^2 + \left(y_0 - \frac{R}{2}\right)^2 = \left(\frac{R}{2}\right)^2$$

Thus, our desired curve is a circle of radius $\frac{R}{2}$ centered at $(0, \frac{R}{2})$.

1.2 Moment of Inertia (MOI)

The rotational analogue of mass is the **moment of inertia,** or the MOI. The MOI depends on the geometry of the shape, which influences its mass distribution.

MOIs are always calculated **about an axis.** Some common MOIs which you should know are:



Remark. By right, you aren't expected to memorise MOIs. However, you are highly encouraged to just memorise them (for these common shapes above), to speed up your working during contests like SPhO! (You'd just need to memorise the numerical prefactors anyway.)

Now, let's look at how to find these MOIs. Interested readers may also consult the Appendix for more interesting but niche methods.

1.2.1 Naive Summation/Integration

In the case of point masses, the MOI is defined as

$$I := \sum_i m_i r_i^2 \tag{12}$$

Thus, the MOI of a **point mass** M at a distance R away from the axis of interest is $I = MR^2$.

However, as usual, most objects are continuous mass distributions instead. As an integral, the MOI is defined as

$$I := \int r^2 \, dm \tag{13}$$

where r is the distance of the infinitesimal mass dm from the axis of interest.

Example 1.3. Determine the MOI of a uniform thin disk of mass M and radius R about its central axis.

We can just imagine the disk to be made of many concentric rings, from radius 0 to radius R. We can then integrate to sum up all the moments of inertia.

For a ring of mass M and radius R, all the mass is equidistant from the central axis, at a distance R away. Thus, we can take out the r^2 from the integral:

$$I_{ring} = \int r^2 \, dm = \int R^2 \, dm = R^2 \int dm = MR^2$$

For the disk, we can define our mass density per unit area as

$$\sigma = \frac{M}{A} = \frac{M}{\pi R^2}$$

Hence, integrating over all the rings,

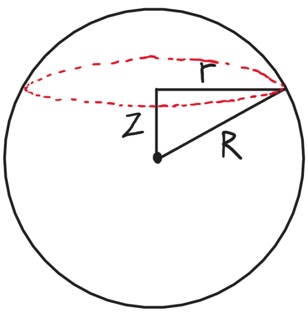
$$I_{disk} = \int dI = \int (dm) \, r^2 = \int_0^R \sigma \, (2\pi r) \, r^2 \, dr = 2\pi\sigma \left[\frac{r^4}{4}\right]_0^R = \frac{\pi \sigma R^4}{2} = \frac{1}{2} MR^2$$

Example 1.4. Determine the MOI of a uniform solid sphere of mass M and radius R about its diameter.

The mass density per unit volume of the sphere is

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Slice the sphere into many disks as such:



We have $r = \sqrt{R^2 - z^2}$, thus $dV = \pi r^2 \, dz = \pi \left(R^2 - z^2\right) \, dz$ for each small disk.

Integrating over all the disks, we get

$$I_{sphere} = \int dI = \int \frac{1}{2} (dm) \, r^2 = \int \frac{1}{2} \left(R^2 - z^2\right) \rho \, dV = \int_{-R}^{R} \frac{3M}{8R^3} \left(R^2 - z^2\right)^2 \, dz = \ldots = \frac{2}{5} MR^2$$

1.2.2 Parallel Axis Theorem

To simplify MOI calculations, the parallel axis theorem states

$$I = I_{CM} + Md^2 \tag{14}$$

where d is the distance between the axis of interest and the axis passing through the CM.

Remark. When applying the parallel axis theorem, one of the axes **must be through the CM.** It is wrong to say that $I_2 = I_1 + Md^2$ for any random points 1 and 2!

Example 1.5. Knowing that the MOI of a thin rod of mass M and length L about the axis through its centre and perpendicular to its length is $I = \frac{1}{12}ML^2$, find the MOI for the same rod about the axis through one of its ends and perpendicular to its length.

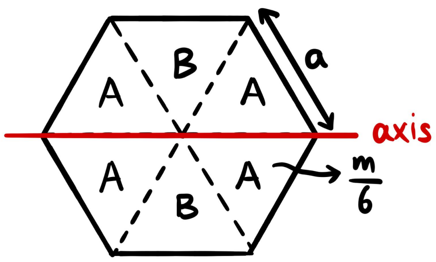
This is a very simple application of the parallel axis theorem:

$$I = I_{CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

Let's look at a more complicated application of the parallel axis theorem.

Example 1.6 (F=ma)**.** The moment of inertia of a uniform equilateral triangle with mass m and side length a about an axis through one of its sides and parallel to that side is $\frac{1}{8}ma^2$. **Using this result,** calculate the moment of inertia of a uniform regular hexagon of mass m and side length a about an axis through two opposite vertices.

Of course, you could have performed naive integration to solve this. However, as the question wants you to use the result of the MOI of the equilateral triangle, we shall split the hexagon up into six equilateral triangles as follows:

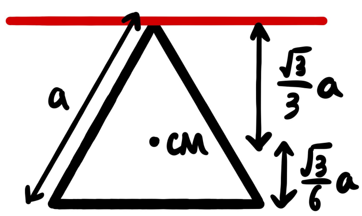


Triangles labelled A and B contribute different MOIs about the axis.

For A, we can directly use the result, so each triangle contributes

$$I_A = \frac{1}{8}\left(\frac{m}{6}\right)a^2 = \frac{1}{48}ma^2$$

For B, we need to use the parallel axis theorem.



By the parallel axis theorem,

$$I_{CM} = I - M\left(\frac{\sqrt{3}}{6}a\right)^2 = \frac{1}{8}\left(\frac{m}{6}\right)a^2 - \frac{m}{6}\left(\frac{1}{12}a^2\right) = \frac{1}{144}ma^2$$

and by the parallel axis theorem again,

$$I_B = I_{CM} + M\left(\frac{\sqrt{3}}{3}a\right)^2 = \frac{1}{144}ma^2 + \frac{m}{6}\left(\frac{1}{3}a^2\right) = \frac{1}{16}ma^2$$

Thus, the final MOI of the hexagon is

$$I_{hexagon} = 4I_A + 2I_B = 4\left(\frac{1}{48}ma^2\right) + 2\left(\frac{1}{16}ma^2\right) = \frac{5}{24}ma^2$$

Remark. When calculating the MOI of B, we applied the parallel axis theorem twice. The first time we applied it, we actually applied it *in reverse,* by finding I_{CM} from I! This trick is very useful for simplifying calculations.

1.2.3 Perpendicular Axis Theorem

Only for planar objects (i.e. objects in 2D), the perpendicular axis theorem states

$$I_z = I_x + I_y \tag{15}$$

Usually, in Physics Olympiad, you'll only apply this for symmetrical situations where $I_x = I_y$.

Example 1.7. Knowing that the MOI of a uniform thin disk of mass M and radius R is $\frac{1}{2}MR^2$ about its central axis, find the MOI of the same disk about its diameter.

This is a very simple application of the perpendicular axis theorem.

By symmetry, $I_x = I_y$, and $I_z = \frac{1}{2}MR^2$. Thus,

$$I_x = I_y = \frac{1}{2}I_z = \frac{1}{4}MR^2$$

where the subscripts indicate the axis which the MOI is calculated with.

1.3 Rotational Dynamics

Now that we understand the kinematics, let's now analyse how rotation even comes about.

1.3.1 Torque and Angular Momentum

The **torque**, $\boldsymbol{\tau}$, is the rotational analogue of force. It is defined as

$$\boldsymbol{\tau} \coloneqq \mathbf{r} \times \mathbf{F} \tag{16}$$

where $\mathbf{r}$ is the position vector (from the origin) of the **point of application** of $\mathbf{F}$.

The **angular momentum**, $\mathbf{L}$, is the rotational analogue of linear momentum. It is defined as

$$\mathbf{L} \coloneqq \mathbf{r} \times \mathbf{p} \tag{17}$$

where $\mathbf{p}$ is the linear momentum of the object.

Using $\mathbf{F} = m\mathbf{a}$ and $\mathbf{p} = m\mathbf{v}$, Equations (16) and (17) are related by

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{18}$$

From Equation (18), you can see that

$$\boldsymbol{\tau} = \mathbf{0} \quad \Rightarrow \quad \mathbf{L} = \text{constant} \tag{19}$$

This is the **conservation of angular momentum,** or COAM. When there is no net torque about a given axis, the angular momentum is conserved!

1.3.2 Rotational Analogue of N2L (Euler's Rotation Equation)

For a **fixed-axis rotation** (i.e. rotation about a stationary axis), we have

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = \mathbf{r} \times (m\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = mr^2\boldsymbol{\omega} = I\boldsymbol{\omega} \tag{20}$$

where the vector triple product and the fact that $\boldsymbol{\omega} \perp \mathbf{r}$ was invoked in the second last equality.

Thus, substituting this into Equation (18),

$$\boldsymbol{\tau} = \frac{d}{dt}(I\boldsymbol{\omega}) = I\boldsymbol{\alpha} \tag{21}$$

Equation (21) is commonly referred to as **Euler's Rotation Equation,** and can be seen as a rotational analogue of N2L (by swapping m with I and $\mathbf{a}$ with $\boldsymbol{\alpha}$).

Remark. When applying Equation (21), it is crucial that you consider the **same pivot/axis** for calculating both your torque and MOI!

You'll usually use N2L to write equation(s) involving forces, and Equation (21) to write an equation involving torques, and solve the equations. Let's see an example on how to apply this.

Example 1.8 (SPOT TST 2022)**.** Consider a yo-yo made up of **two** uniform solid disks of radius R and each with mass M, connected rigidly by a light cylindrical axle of radius $r < R$, such that the disks and axle all share a common axis. A thin light string is wound tightly around the axle. The free end of the string is held fixed and the yo-yo is released from rest. Assume that the string stays vertical as it is unwound from the axle. (i) Determine the downward acceleration a of the yo-yo when it is released from rest. (ii) Determine the tension T in the string when the yo-yo is released from rest.

Firstly, the MOI of the yo-yo about its central axis is given by

$$I = 2I_{disk} = 2\left(\frac{1}{2}MR^2\right) = MR^2$$

We can write N2L for the translational motion of the yo-yo:

$$2Mg - T = 2Ma$$

Taking torques about the centre of the yo-yo, we can write Euler's Rotation Equation for the rotational motion of the yo-yo:

$$Tr = I\alpha = MR^2\alpha$$

The last equation we need is the non-slip condition to relate a and α. Since the ICOR in this case is the point on the yo-yo where the string is attached (at a distance r away from the centre), the non-slip condition is

$$a = r\alpha$$

Combining all the results above, you should eventually get

$$a = \frac{2r^2}{2r^2 + R^2}g, \ T = \frac{2MR^2}{2r^2 + R^2}g$$

1.3.3 Rotational Kinetic Energy, Work and Power

When objects rotate, there is a kinetic energy associated with rotation.

Since there is a kinetic energy of $K_{trans} = \frac{1}{2}mv^2$ associated with translation, by taking the rotational analogues of m and v, the **rotational kinetic energy** is defined as

$$K_{rot} := \frac{1}{2}I\omega^2 \tag{22}$$

More generally, for an object that is rolling, we have

$$K = K_{trans} + K_{rot} = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \tag{23}$$

Remark. To avoid confusion, always consider I and v **of the CM** when calculating total KE!

For a rolling body, the **instantaneous power** by a force $\mathbf{F}$ causing a torque $\boldsymbol{\tau}$ is

$$P_{instantaneous} = \mathbf{F} \cdot \mathbf{v}_{CM} + \boldsymbol{\tau}_{CM} \cdot \omega \tag{24}$$

and the work done is

$$W = \int P_{instantaneous}\, dt = \int (\mathbf{F} \cdot d\mathbf{R} + \boldsymbol{\tau}_{CM} \cdot d\boldsymbol{\theta}) \tag{25}$$

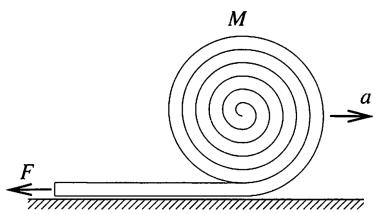
In Equations (24) and (25), the first term is referred to as **translational work/power** and the second term is referred to as **rotational work/power.** Don't forget the rotational work/power!

In rotation, the **work-energy theorem** also holds if we account for K_{rot}. We have

$$W_{total} = \Delta K = \Delta K_{trans} + \Delta K_{rot} \tag{26}$$

Now, let's look at an example where we must consider the rotational kinetic energy.

Example 1.9 (200 Puzzling Physics Problems)**.** A fire hose of mass M and length L is coiled into a roll of radius R. The hose is sent rolling along level ground, with its centre of mass given initial speed $v_0 >> \sqrt{gR}$. The free end of the hose is held fixed. The hose unrolls and becomes straight. How long does this process take to complete?



First, let's rationalise what $v_0 >> \sqrt{gR}$ means. Since v_0 is large, this simply means that changes in GPE are negligible compared to KE. Thus, when writing COE, we may drop all GPE terms.

Thus, accounting for the rotational kinetic energy, the COE equation is

$$K_{trans,i} + K_{rot,i} = K_{trans,f} + K_{rot,f} \quad \Rightarrow \quad \frac{1}{2}Mv_0^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2$$

The non-slip conditions give $v_0 = R\omega_0$ and $v = r\omega$, hence we can simplify the above equation:

$$\frac{3}{4}Mv_0^2 = \frac{3}{4}mv^2 \quad \Rightarrow \quad Mv_0^2 = mv^2 \quad \Rightarrow \quad v = \sqrt{\frac{M}{m}}v_0$$

Here, m is the mass of the portion of the hose that is still rolling. After the hose travels a distance x, by proportionality, we have

$$m(x) = M\left(1 - \frac{x}{L}\right)$$

Substituting this into the expression for v, we have

$$v = \frac{dx}{dt} = \frac{v_0}{\sqrt{1-\frac{x}{L}}} \quad \Rightarrow \quad dt = \frac{1}{v_0}\sqrt{1-\frac{x}{L}}\,dx \quad \Rightarrow \quad \int_0^t dt = \frac{1}{v_0}\int_0^L \sqrt{1-\frac{x}{L}}\,dx \quad \Rightarrow \quad t = \frac{2L}{3v_0}$$

Remark. You might ask - why is *energy* conserved, if the hose is inelastically sticking to the ground as it unrolls? This process actually dissipates no energy, because the circular part of the hose is rolling *without* slipping, so the bottom part is the ICOR and has no velocity, hence no work is done against friction!

1.3.4 Rotational Collisions

A useful quantity in rotational collisions is the **angular impulse, $\mathbf{J}_\theta$**. (There is actually no universal symbol for it.) It is the rotational analogue of linear impulse, and is defined as

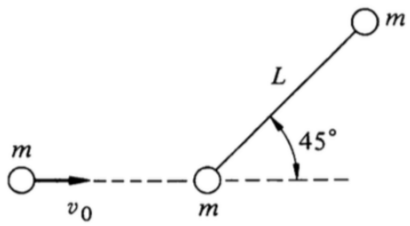
$$\mathbf{J}_\theta = \int \boldsymbol{\tau}\,dt \tag{27}$$

There also exists the **angular impulse-momentum theorem:**

$$\mathbf{J}_\theta = \Delta\mathbf{L} \tag{28}$$

In rotational collisions, we usually deal with translation and rotation separately, to write separate equations for COM and COAM. If the collision is elastic, we can also write COE.

Example 1.10 (Kleppner & Kolenkow)**.** A rigid, massless rod of length L joins two particles, each of mass m. The rod lies on a frictionless table and is struck by a particle of mass m and velocity v_0 as shown. After an **elastic** collision, the particle moves **straight back.** Find the angular velocity of the rod about its CM after the collision.



Firstly, the MOI of the rod + 2 particles about its CM is

$$I = 2\left(m\left(\frac{L}{2}\right)^2\right) = \frac{1}{2}mL^2$$

Many quantities are conserved here. Suppose the projectile moves backwards with speed v_1, the CM speed of the rod is v_2, and its angular velocity about its CM is ω.

By COM, we have

$$mv_0 = -mv_1 + 2mv_2 \quad \Rightarrow \quad v_0 + v_1 = 2v_2$$

By COAM about the axis parallel to v_0 and passing through the initial CM of the rod, we have

$$\frac{mv_0L}{2\sqrt{2}} = \frac{1}{2}mL^2\omega - \frac{mv_1L}{2\sqrt{2}} \quad \Rightarrow \quad v_0 + v_1 = \sqrt{2}L\omega$$

By COE, we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + 2\left(\frac{1}{2}mv_2^2\right) + \frac{1}{2}\left(\frac{1}{2}mL^2\right)\omega^2 \quad \Rightarrow \quad (v_0 - v_1)(v_0 + v_1) = 3v_2^2$$

The COM and COE equations should be relatively simple to write. However, for COAM, you need to be careful of the signs of each term! You should go back to Equations (17) and (20) to determine the direction of the vectors, and assign different signs to different directions.

Now, solving the three simultaneous equations above, we get $v_2 = \frac{4}{7}v_0$, and hence

$$\omega = \frac{\sqrt{2}v_2}{L} = \frac{4\sqrt{2}}{7}\frac{v_0}{L}$$

Remark. You might ask - why do we take COAM about *this* axis? Why not take it, for instance, about the point of collision? Actually, both are equally valid! However, the bodies that should be in your system are different (to ensure $\boldsymbol{\tau} = \mathbf{0}$ on your system):

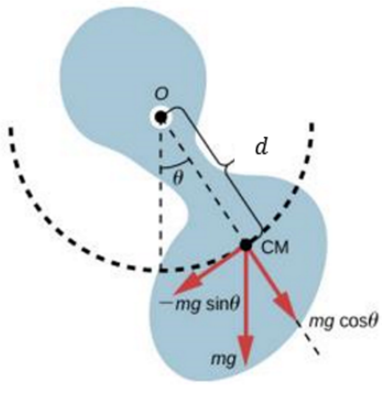
1. **About the centre of the rod:** Both the moving particle and the rod + 2 particles
2. **About the collision point:** Only the rod + 2 particles

1.3.5 Physical Pendulums

In oscillations, you learnt that for a simple pendulum,

$$\omega_{simple} = \sqrt{\frac{g}{L}}, \; T_{simple} = 2\pi\sqrt{\frac{L}{g}} \tag{29}$$

However, this only applies if the pendulum bob is a point mass, and if the string connecting the pivot to the pendulum bob is light! Instead, for a general rigid body swinging freely about some pivot point, we have a **physical pendulum.**



We can use Equation (21) on the torque due to gravity. Assuming $\theta << 1$,

$$mgd\sin\theta = -I_{pivot}\ddot{\theta} \quad \Rightarrow \quad \ddot{\theta} + \frac{mgd}{I_{pivot}}\theta = 0 \tag{30}$$

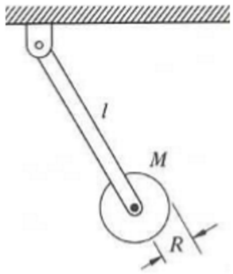
where we take torques and calculate the MOI about the pivot O. The negative sign is essential as the torque due to gravity decreases θ.

This is the form of a SHM, with

$$\omega_{physical} = \sqrt{\frac{mgd}{I_{pivot}}}, \ T_{physical} = 2\pi\sqrt{\frac{I_{pivot}}{mgd}} \tag{31}$$

Using the parallel axis theorem, you can also write $I_{pivot} = I_{CM} + md^2$ to simplify this.

Example 1.11 (Ricardo)**.** A physical pendulum consists of a disk of radius R and mass m_d, fixed at the end of a rod of mass m_r and length l. (i) Find the period of the pendulum. (ii) How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is free to spin?



(i) The MOI of the physical pendulum about its pivot is

$$I_{pivot} = I_{rod} + I_{disk} = \frac{1}{3}m_r l^2 + \left(\frac{1}{2}m_d R^2 + m_d l^2\right)$$

where we have used the parallel axis theorem for the MOI of the disk.

Let θ be the angle of the rod with respect to the vertical. As usual, we assume $\theta << 1$. The net torque due to gravity about the pivot is

$$\tau = \left(\frac{1}{2}m_r + m_d\right) gl\theta$$

Thus, being careful with signs, we write Euler's Rotation Equation:

$$\left(\frac{1}{3}m_r l^2 + \frac{1}{2}m_d R^2 + m_d l^2\right)\ddot{\theta} = -\left(\frac{1}{2}m_r + m_d\right) gl\theta \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{\frac{1}{3}m_r l^2 + \frac{1}{2}m_d R^2 + m_d l^2}{\left(\frac{1}{2}m_r + m_d\right) gl}}$$

(ii) The difference here is that since the disk is free to rotate, we lose the $\frac{1}{2}m_d R^2$ term. In other words, we are essentially treating the disk as a point mass now.

With this correction, we instead have

$$T = 2\pi\sqrt{\frac{\frac{1}{3}m_r l^2 + m_d l^2}{\left(\frac{1}{2}m_r + m_d\right) gl}} = 2\pi\sqrt{\frac{\left(\frac{1}{3}m_r + m_d\right) l}{\left(\frac{1}{2}m_r + m_d\right) g}}$$

Remark. This example shows you a subtlety in calculating MOI. When we add up MOIs like we did in (i), we must ensure that all the components we are adding up are **rigidly connected** (i.e. we can treat the whole body as one rigid body).

1.4 Ideas

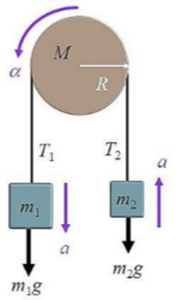
Many tricky mechanics problems involve the use of the following ideas.

1.4.1 Massive Pulleys

In dynamics, we always made an assumption that all pulleys were massless (and small). However, that is not the case in real life!

When we consider rotation, we usually model pulleys as **uniform disks** of finite mass and radius. Consequently, the net force and torque on them need not be zero!

Example 1.12. Consider a pulley system with masses m_1 and m_2, with $m_2 < m_1$. The mass of the pulley is M and its radius is R. Assuming the pulley and string doesn't slip, determine the acceleration of the masses.



The first, most important thing for you to realise is that the **tension in the string is not constant!** This allows for a non-zero net torque on the pulley, allowing it to rotate.

Thus, writing N2L for each mass,

$$m_1 g - T_1 = m_1 a,\ T_2 - m_2 g = m_2 a$$

and taking torques about the centre of the pulley,

$$(T_1 - T_2) R = \frac{1}{2} M R^2 \alpha$$

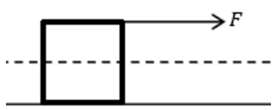
Combining with the non-slip condition $a = R\alpha$, we eventually get

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M} g$$

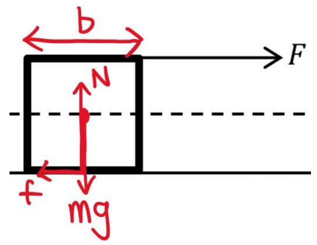
1.4.2 Where Does Normal Force Act?

When it comes to rigid bodies of finite size, there can be some confusion on where exactly the normal force acts. The example below illustrates this.

Example 1.13 (Ricardo)**.** Consider a solid cube of mass m and dimensions $b \times b \times b$ moving linearly on the rough horizontal ground with initial horizontal velocity v_0. An external force F acts in the same direction as the initial velocity at its edge, as shown below. The coefficient of friction between the cube and the ground is μ. Determine the maximum value of F before the cube topples.

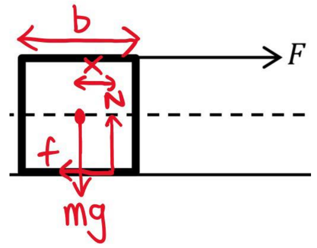


Intuitively, you might draw the FBD of the cube as such:



However, this makes no sense! If we take torques about the CM, then the forces mg and N contribute no torque, while F and f both contribute clockwise torques. This would imply that any value of F, no matter how small, would cause a non-zero clockwise torque about this point, causing the tube to topple!

The **correct FBD** considers a different point of action for N. Suppose that the point of action of N is shifted by some distance x (to the right, so that the torque by N opposes the torque by F and f).



From N2L, we know that

$$N = mg, \; F - \mu N = ma$$

Now, taking torques about the CM,

$$Nx = \frac{(F+f)\,b}{2} \quad \Rightarrow \quad mgx = \frac{(F+\mu mg)\,b}{2} \quad \Rightarrow \quad x = \left(\frac{F}{mg} + \mu\right)\frac{b}{2}$$

Clearly, the point of action of N must still remain in the cube. Thus,

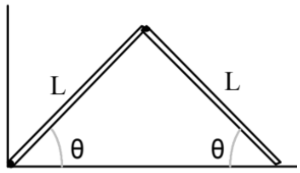
$$x \leq \frac{b}{2} \quad \Rightarrow \quad F \leq mg\,(1-\mu) \quad \Rightarrow \quad F_{max} = mg\,(1-\mu)$$

Remark. This example shows you that if you wrongly assume the position at which your normal force acts, you may get nonsensical results (in this case, it led us to conclude that even the smallest force would lead to the cube toppling, which we know is not true from daily life). In such a case, rethink where the normal force should act!

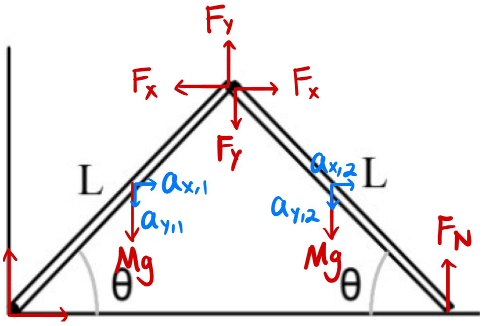
1.4.3 Inextensibility Constraints

When dealing with rotation problems involving **rods,** the inextensibility (or rigidity) of the rods gives rise to an additional constraint equation.

Example 1.14 (USAPhO 1999)**.** Consider two uniform rods each of mass M and length L in a "V" shape, whose angle can vary. The rods are released from rest when their angle θ to the horizontal is 45°. All hinges are frictionless and have negligible mass. The left end is fixed, and the right end is free to slide on the horizontal surface without friction. Find the upward force F_N that the horizontal surface exerts on the right end just after release.



We first label all forces with red and all accelerations with blue:



The motion of the 2 rods will be symmetric (we don't expect the angles θ to be different). To write our constraint equation, define the fixed left end as the origin. Then,

$$x_{CM,2} = 3x_{CM,1} \quad \Rightarrow \quad a_{x,2} = 3a_{x,1}$$

$$y_{CM,1} = y_{CM,2} \quad \Rightarrow \quad a_{y,1} = a_{y,2}$$

by geometry and then differentiating with respect to time. These are the constraint equations.

Additionally for the left rod, as $\theta = 45°$, the symmetry gives that

$$a_{x,1} = a_{y,1}$$

Also, the magnitude of a is

$$a = \sqrt{a_{x,1}^2 + a_{y,1}^2} = \frac{\alpha L}{2} \quad \Rightarrow \quad a_{x,1} = a_{y,1} = \frac{\alpha L}{2\sqrt{2}}$$

Now, we can proceed with writing N2L for the right rod:

$$F_x = Ma_{x,2} = M\frac{3\alpha L}{2\sqrt{2}}, \ Mg + F_y - F_N = Ma_{y,2} = M\frac{\alpha L}{2\sqrt{2}}$$

Taking torques about the hinge at the fixed left end of the left rod,

$$\frac{1}{3}ML^2\alpha = \frac{MgL}{2\sqrt{2}} - \frac{F_xL}{\sqrt{2}} - \frac{F_yL}{\sqrt{2}}$$

Taking torques about the CM of the right rod,

$$\frac{1}{12}ML^2\alpha = \frac{F_NL}{2\sqrt{2}} + \frac{F_yL}{2\sqrt{2}} - \frac{F_xL}{2\sqrt{2}}$$

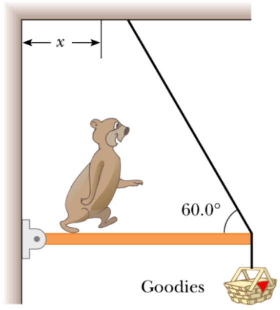
We can solve these simultaneous equations to eventually get

$$F_N = \frac{7}{10}Mg$$

2 Problems

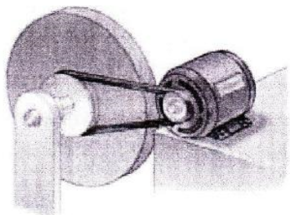
Problems are arranged in roughly increasing difficulty.

Problem 2.1 (SPhO 2008)**.** A hungry bear weighing 700 N walks out on a beam in an attempt to retrieve a basket of food hanging at the end of the beam. The beam is uniform, weighs 200 N, and is 6.00 m long; the basket weighs 80.0 N. (i) When the bear is at $x = 1.00$ m, find the tension in the wire and the components of the force exerted by the wall on the left end of the beam. (ii) If the wire can withstand a maximum tension of 900 N, what is the maximum distance the bear can walk before the wire breaks?

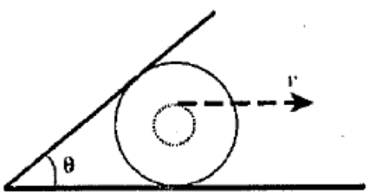


Problem 2.2 (SPhO 2005)**.** A 15.0 m uniform ladder of mass 50 kg rests against a frictionless wall and makes an angle of 60° to the horizontal. (i) Determine the horizontal and vertical forces that the ground exerts on the base of the ladder when an 800 N firefighter is 4.00 m along the ladder from the bottom. (ii) If the ladder is just on the verge of slipping when the firefighter is 9.00 m up, what is the coefficient of static friction between the ladder and the ground?

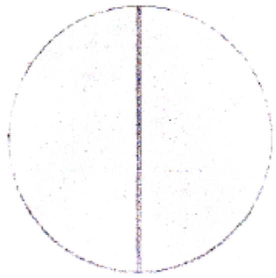
Problem 2.3 (SPhO 2010)**.** An electric motor turns a flywheel through a drive belt that joins a pulley on the motor and a pulley that is rigidly attached to the flywheel, as shown in the figure. The flywheel is a solid disk with a mass of 80.0 kg and a diameter of 1.25 m. It turns on a frictionless axle. Its pulley has a much smaller mass and a radius of 0.230 m. If the tension in the upper, taut segment of the belt is 135 N and the flywheel has a clockwise angular acceleration of 1.67 rad s^{-2}, find the tension in the lower, slack segment of the belt.



Problem 2.4 (SPhO 2014)**.** A yo-yo is pulled by its string along a horizontal surface without slipping. The horizontal velocity at the end of the string remains equal to v. A bar is pivoted as shown and remains supported by the yo-yo. Find the angular speed of the bar, ω, as a function of the angle θ. The outer and inner radii of the yo-yo are R and r respectively.

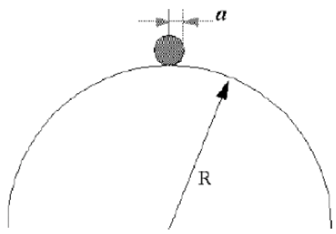


Problem 2.5 (SPhO 2012)**.** A pendulum bob is constructed by taking a thin, uniform density circular ring of mass M and radius R and affixing a straight, thin uniform rod of mass m and length $2R$ across its diameter as shown in the diagram. The pendulum hangs in a vertical plane from a frictionless pivot that can be attached to the ring at any point. The pivot allows the pendulum to swing either in the plane of the bob or in the plane perpendicular to the bob. Assume the angular amplitude is small. (i) What swinging configuration(s) give the maximum period? (ii) What swinging configuration(s) give the minimum period? (iii) Find the ratio of the maximum to the minimum period.

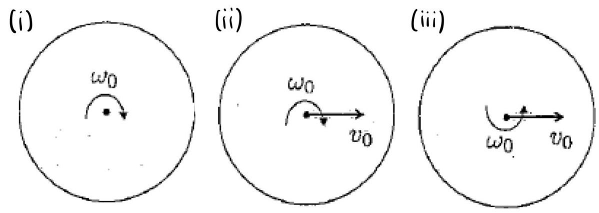


Problem 2.6 (Ricardo)**.** Compute the MOI of a uniform solid cone of mass M, base radius R and height H about (i) its symmetry axis (ii) the diameter of its base.

Problem 2.7 (SPhO 2003)**.** A marble rests on the top of a hemispherical bowl as shown in the figure below. The radii of the bowl and the marble are R and a respectively. The marble is displaced from rest and it rolls down the bowl without slipping. At what height, in terms of R and a, will the marble leave the hemisphere?

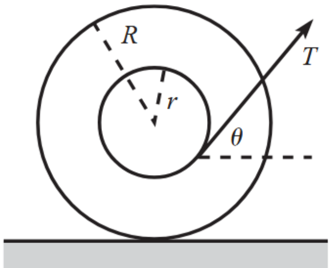


Problem 2.8 (SPhO 2016)**.** Consider a rigid, axially symmetric wheel with mass M, moment of inertia I about the axis of symmetry S, and radius R, moving in an upright position on a horizontal plane. The coefficient of kinetic friction between the wheel and the plane is μ_k. (i) The wheel is initially spun about S with constant angular speed ω_0 and is then released. It slips for time T_a and then rolls without slipping. Determine T_a and the centre of mass speed v_a of the rolling wheel after it stops slipping. (ii) The wheel has an initial horizontal centre of mass speed v_0 in addition to an initial angular speed ω_0. Determine the time T_b at which the wheel rolls without slipping and the centre of mass speed v_b of the rolling wheel. (iii) The wheel has an initial horizontal centre of mass speed v_0 and an initial **backspin** $-\omega_0$. Determine the possible motions of the wheel.

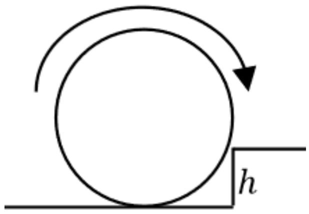


Problem 2.9 (Ricardo)**.** A uniform solid disk of mass M and radius R rotates with initial angular speed ω_0 about its centre while lying on the ground. The coefficient of friction between the disk and the ground is μ. (i) Find the total torque acting on the disk about its centre. (ii) Find the time taken for the disk to stop rotating. (In engineering, this is called disk friction.)

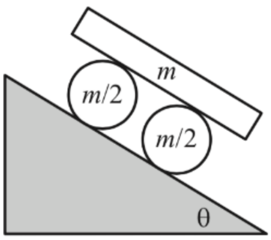
Problem 2.10 (Morin)**.** A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground. A thread is wrapped around the axle and is pulled with tension T at an angle θ to the horizontal. (i) Given R and r, what should θ be so that the spool doesn't move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip. (ii) Given R, r and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest? (iii) Given R and μ, what should r be so that you can make the spool slip from the static position with as small a T as possible? That is, what should r be so that the upper bound on T in (ii) is as small as possible? What is the resulting value of T?



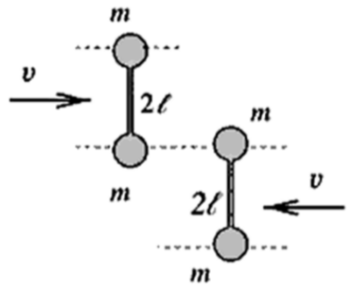
Problem 2.11 (Ricardo)**.** A uniform solid cylinder of mass M and radius R is rolling without slipping on the ground with an angular speed of ω_0. It hits an edge of height h and rolls up. The point of contact does not slip during the up-rolling. (i) Calculate the angular speed of the cylinder immediately after the collision, assuming the collision time is very short. (ii) Calculate the minimum value of ω_0 such that the cylinder is able to roll up. (Hint: For a very short collision time, what quantity is conserved? Why?)



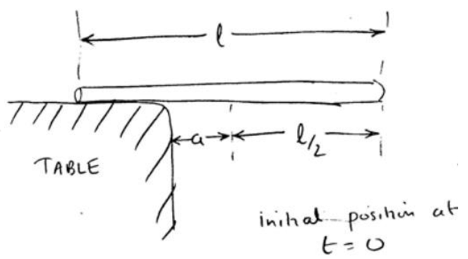
Problem 2.12 (Morin)**.** Consider the following "car" on an inclined plane. It is made out of a board (a thin sheet) of mass m and two cylinders, each of mass $\frac{m}{2}$ and radius R. If it is released from rest, and there is no slipping between any surfaces, find the acceleration of the board. (You should find that the acceleration of the board *could* exceed g if θ can be freely varied between 0° and 90°. Can you rationalise why?)



Problem 2.13 (200 Puzzling Physics Problems)**.** Two identical dumbbells move towards each other on a frictionless table as shown. Each consists of two point masses m joined by a massless rod of length $2l$. The dumbbells collide elastically. (i) Find the angular velocity of the rods right after collision. (ii) Describe the subsequent evolution of the system.



Problem 2.14 (Ricardo)**.** A uniform rod of mass m and length l is placed on a horizontal table top with its CM a distance a from the perpendicular edge. The rod is released from rest from a horizontal position and begins to rotate about the edge of the table. The coefficient of friction between the rod and the table is μ. (i) Find the maximum angle that the rod makes with the horizontal before slipping begins. (ii) Now, let only a quarter of the rod's length be on the table. What angle does the rod make with the horizontal when it starts to slip? The coefficient of static friction between the rod and the table is $\mu = \frac{7}{13}$.

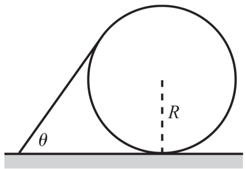


Problem 2.15 (Kleppner & Kolenkow)**.** A uniform plank of length $2L$ leans nearly vertically against a wall. The wall and the ground are frictionless. The plank starts to slip downwards. Find the height of the top of the plank when it loses contact for the first time (either the wall or the floor).

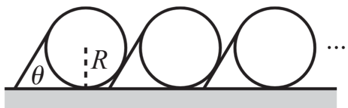
3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

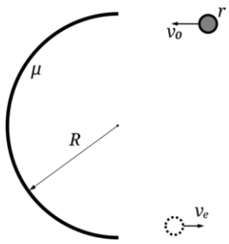
Problem 3.1 (SPhO 2011, Morin). (i) A stick of mass density per unit length ρ rests on a circle of radius R. The stick makes an angle θ with the horizontal and is tangent to the circle at its upper end. Friction exists at all points of contact, and assume that it is large enough to keep the system at rest. Find the friction force between the ground and the circle.



(ii) A large number of identical sticks and circles lean on each other, as shown in the figure. Each stick makes an angle θ with the horizontal and is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is frictionless (unlike in (i)). What is the normal force between a stick and the circle it rests on, very far to the right? (Assume the last circle is fixed and cannot move.)

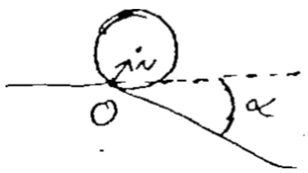


Problem 3.2 (EuPhO 2024). A puck (a small disc) with radius r and uniform density is moving on a horizontal plane with the velocity v_0 **without rotation.** The puck meets a **fixed** half-circular wall of radius $R >> r$ and starts to move along the wall. The coefficient of friction with the wall is μ, and friction with the horizontal plane is negligible. (i) Find the velocity of the puck v_e when it leaves the wall. (ii) Sketch the graph of $v_e(\mu)$. Indicate all important features.



While this is a EuPhO problem, don't be intimidated! Many competitors received full credit on this problem (T1) that year, making it easy for EuPhO standards.

Problem 3.3 (Ricardo). A solid cylinder of radius R rolls without slipping with CM speed v_0 on a horizontal ground. At point O, it rotates with point O as the rotational axis, and continues rolling without slipping on an inclined plane of angle α afterwards. Determine the maximum value of v_0 such that the cylinder does not lose contact with point O.



4 Appendix

4.1 More Methods of Calculating MOI

There are many elegant ways you can calculate MOI, other than the methods introduced in Section 1.2. Here are a few research papers you can read to find out more!

1. Scaling Method
2. Squashing Method
3. Three-Axis Theorem
4. Analysis of Differential Elements

While you don't need to know all of these methods for Physics Olympiad, hopefully these give you some insight into how different mass distributions from a given axis influence the MOI!